Dependence of frequency of nonlinear cold plasma cylindrical oscillations on electron density

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Dependence of frequency of nonlinear cold plasma cylindrical oscillations on electron density

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For cold plasma, the frequency of small amplitude Langmuir oscillations along one Cartesian coordinate is $\omega_{po}=\sqrt{4\pi e^2 n_o/m_e}$, where $n_p=n_o$ is the constant proton density which is equal to the average electron density $\overline{n_e}$. This formula for ω_{po} is the basis for measurements of $\overline{n_e}$ in passive and active radio experiments located on spacecraft. We find that for cold plasma nonlinear cylindrical oscillations $\overline{n_e}>n_p$ (i.e., a buildup of negative space charge near the axis of the cylinder). The resulting frequency of oscillations ω_{pe} is greater than ω_{po}. The relation between ω_{pe} and $\overline{n_e}$ is found to be logarithmic: $\omega_{pe}=\omega_{po}[1+\ln(\overline{n_e}/n_o)/12]$ with 0.5% accuracy for the range $1\leqslant \overline{n_e}/n_p<6$. For quasi-neutral plasma, when $\overline{n_e}/n_p\approx 1$, the logarithmic formula reduces to the linear one: $\omega_{pe}=\omega_{po}[11/12+(\overline{n_e}/n_p)/12]$. For $\overline{n_e}/n_p\gg 6$, ω_{pe} approaches an upper limit of $\sqrt{2}\omega_{po}$. These results are expected to be helpful in diagnostics of $\overline{n_e}$ in the solar wind and in magnetospheric plasmas as well as in laboratory plasmas where cylindrical symmetry is present. [DOI: 10.1063/1.1690299]

Cold plasma electrostatic oscillations can be described within a single-fluid approximation where the proton density n_p is considered to be constant. The corresponding equations (cgs units) are the momentum equation

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{e}{m_e}\mathbf{E}, \tag{1}$$

the continuity equation

$$\frac{\partial n_e}{\partial t}+\nabla\cdot n_e\mathbf{v}=0, \tag{2}$$

and Poisson's equation

$$\nabla\cdot\mathbf{D}=-4\pi e(n_e-n_p), \tag{3}$$

where $\mathbf{D}=\epsilon\mathbf{E}$ and $\epsilon=1$. For oscillations along one Cartesian coordinate (plane oscillations), Eqs. (1)–(3) have been reduced to a single equation for a linear oscillator with Langmuir frequency

$$\omega_{po}=\sqrt{4\pi e^2 n_p/m_e}, \tag{4}$$

where $n_p=n_o=\text{const.}$[1–5] For plane oscillations, for any amplitude, $\overline{n_e}=n_p$, where $\overline{n_e}$ is the electron density averaged over a period of harmonic oscillations. Recent papers[6–9] on nonlinear Langmuir waves contain extensive references to work done within kinetic theory. Non-neutral plasmas have been studied extensively (see Refs. 10 and 11 and references therein). Eigenmode solutions confirm again that the frequency of the fundamental Langmuir component ($n=1$) in the cold plasma limit ($T_e\to 0$) does not shift from the original Langmuir value ω_{po} for plane geometry (formula 18 in Ref. 7).

The same system Eqs. (1)–(3) for cylindrical self-similar oscillations can be reduced to a second order nonlinear ordinary differential equation for the evolution function $Y(\tau)$ (basic equation)

$$\frac{d^2Y}{d\tau^2}+Y-Y^{-1}=0, \tag{5}$$

where

$$\tau\equiv t\omega_{po}/\sqrt{2}, \tag{6}$$

$$n_e(\tau)=Y^{-2}(\tau)n_p, \tag{7}$$

$$Y(\tau)=y(\tau)/y_{eq}, \tag{8}$$

$$\mathbf{v}=\eta Y'(\tau)\frac{\omega_{po}}{\sqrt{2}}y_{eq}\mathbf{e_r}, \tag{9}$$

$$\mathbf{E}=2\pi e n_p\eta(Y-Y^{-1})y_{eq}\mathbf{e_r}. \tag{10}$$

Self-similar cylindrical oscillations which include oscillations of the magnetic field have been studied by the authors[12] previously. Equation (5) corresponds to the case without magnetic field ($\Omega_H=b=0$ in notation of Ref. 12). This equation has been derived by a different method[13] with somewhat different notation (function ρ instead of Y). By substituting $X=1+\rho$ in Eq. (12) from Ref. 13, one arrives at our Eq. (5).

The self-similar parameter η is defined as

$$\eta=r/y(t), \tag{11}$$

where r is the distance from the axis of symmetry in a cylindrical system of coordinates. At equilibrium, $y(t)=y_{eq}$ (i.e., $Y_{eq}=1$). In the solar wind, where ω_{po}/ω_{ce} is of the order of 100 (ω_{ce} is the electron gyrofrequency), Eq. (4) of Tonks and Langmuir[3] is used to find n_p. Under the assumption $\overline{n_e}=n_p$, the identification of ω_{po} in the observed spectra leads to the determination of $\overline{n_e}$, namely,

$$\overline{n_e}=\frac{\omega_{po}^2 m_e}{4\pi e^2}. \tag{12}$$

The validity of this formula for large amplitude cylindrical oscillations is the subject of our paper. The comparison of cylindrical, spherical and plane Langmuir oscillations of cold plasma demonstrated that, in contrast to the plane geometry case, cylindrical oscillations are inherently nonlinear with electron frequency ω_{pe} shifted upwards with increase of oscillation amplitude.[13,14] In the following, we explore the impact of this nonlinearity on the relation between $\overline{n_e}$ and ω_{pe}.

Equation (5) is the equation for a nonlinear oscillator with effective potential

$$U(Y)=Y^2/2-\ln Y. \tag{13}$$

Near the minimum of U (i.e., near $Y_{eq}=1$), we can linearize Eq. (5). Taking $Y=1+\Delta Y$, where $|\Delta Y|\ll 1$ (see Appendix and also Ref. 13), we obtain the harmonic oscillator equation $\Delta Y''=-2\Delta Y$, with the nondimensional frequency $\sqrt{2}$. This frequency corresponds to the Langmuir frequency ω_{po} according to Eq. (6). Therefore, for small amplitude cylindrical oscillations, $\overline{n_e}=n_p$ and Eq. (12) is valid. However, this is not true for the large amplitude case. On solving numerically the basic equation (5) and then averaging $n_e(\tau)$, defined by Eq. (7) over one period of nonlinear oscillations, we have obtained the dependence of ω_{pe}/ω_{po} on $\overline{n_e}/n_p$. This dependence is shown in Fig. 1(a) by the open points. The continuous line represents our approximate formula

$$\frac{\omega_{pe}}{\omega_{po}}=1+\frac{1}{12}\ln\left(\frac{\overline{n_e}}{n_o}\right) \tag{14}$$

which fits the numerical solution to within 0.5% over the range $1\leqslant\overline{n_e}/n_o<6$. Figure 1(b) illustrates the deviation of the numerical values of ω_{pe}/ω_{po} from formula (14).

For $|\overline{n_e}/n_p-1|\ll 1$, Eq. (14) reduces to

$$\frac{\omega_{pe}}{\omega_{po}}=\frac{11}{12}+\frac{1}{12}\frac{\overline{n_e}}{n_p}. \tag{15}$$

For this case of weak nonlinearity, the following formulas are true:

$$\frac{\omega_{pe}}{\omega_{po}}=1+\frac{1}{12}a^2. \tag{16}$$

$$\frac{\overline{n_e}}{n_o}=1+a^2, \tag{17}$$

where $a\equiv Y(0)-Y_{eq}=Y(0)-1$. Formula (16) for the upward shift of ω_{pe} has been derived from the third order expansion of the nonlinear force in the basic equation (5) near the equilibrium point $Y_{eq}=1$ (see the Appendix). Equation (15) can also be obtained from Eqs. (16) and (17). For $\overline{n_e}/n_p\geqslant 6$ (i.e., $a\gg 1$), the approximation (14) is not valid. For $a\gg 1$ the effective potential (13) is equivalent to the harmonic potential $Y^2/2$ with a vertical wall near $Y=0$. Such a potential has a frequency $\omega=\sqrt{2}\omega_{po}$.[14] The numerical solution for the weakly nonlinear case $[a=0.05,\ Y'(0)=0]$ is shown in Fig. 2. The solution $Y(\tau)$ is sinusoidal [Fig. 2(a)] and the corresponding phase diagram (i.e., the dependence of Y' on Y) is an ellipse [Fig. 2(b)]. For this case, $\omega_{pe}/\omega_{po}=1.000\,21$ and $\overline{n_e}/n_p=1.002\,46$. By comparison, Eq. (14) for $\overline{n_e}/n_p=1.002\,46$ gives $\omega_{pe}/\omega_{po}=1.000\,204\,7$ while Eq. (15) yields $\omega_{pe}/\omega_{po}=1.000\,205$. For the nonlinear case $a=1.5$, $Y'(0)=0$ shown in Figs. 3(a) and 3(b), the corresponding solution is not sinusoidal and the corresponding phase diagram is not ellipsoidal. The exact numerical values are $\omega_{pe}/\omega_{po}=1.141\,82$, $\overline{n_e}/n_p=5.910\,93$. By comparison, Eq. (14) yields $\omega_{pe}/\omega_{po}=1.148\,07$.

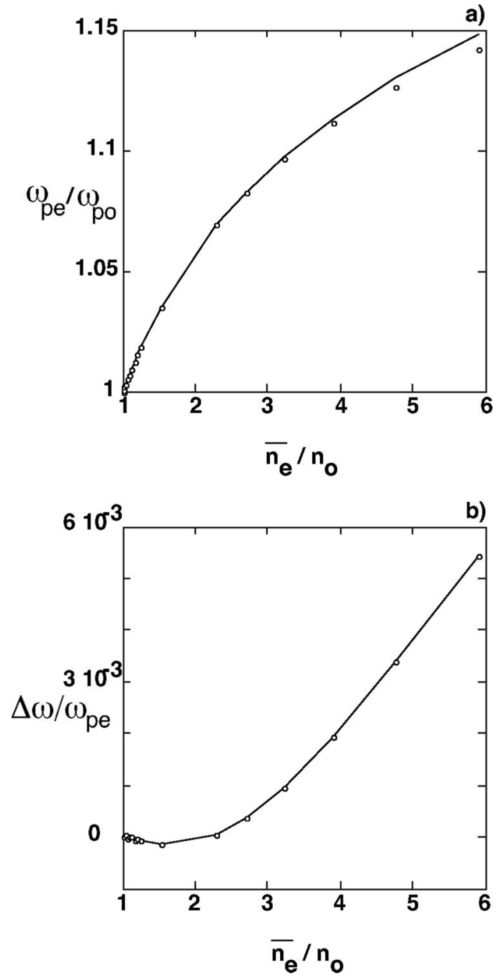

FIG. 1. (a) Dependence of ω_{pe}/ω_{po} on normalized average electron density $\overline{n_e}/n_o$. Numerical results are presented as open circles. Continuous line represents approximate logarithmic fit Eq. (14). (b) Deviation of analytic approximation Eq. (14) from numerical values of ω_{pe}/ω_{po} for the range of $1\leqslant\overline{n_e}/n_o<6$. $\Delta\omega\equiv(\omega_{pe})_{analytic}-(\omega_{pe})_{numeric}$.

Plane Langmuir cold plasma oscillations according to Akhiezer and Lyubarskii[1] remain harmonic with the frequency ω_{po} for any amplitude. For such oscillations there is no buildup of space charge, i.e., $\overline{n_e}=n_p$ where n_p is assumed to be constant $n_p=n_o$. Our work shows this is not the case for cylindrical oscillations. In this case, in addition to the upward frequency shift $\omega_{pe}>\omega_{po}$, there is an upward shift

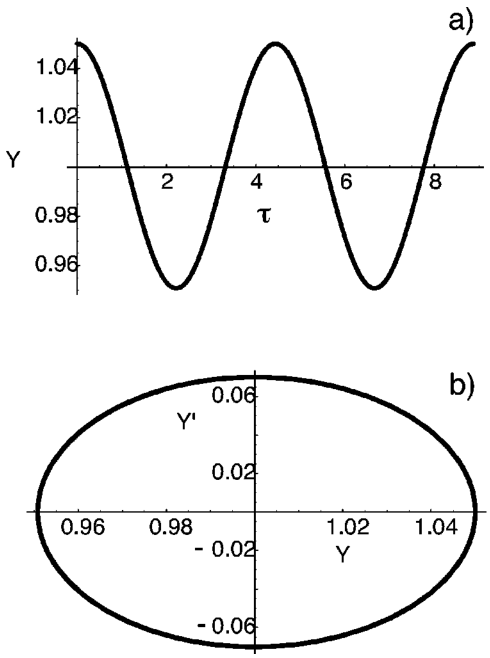

FIG. 2. (a) Numerical solution of the basic equation (5) for the weakly nonlinear case of $a=0.05$ ($Y_{\max}=1.05$). (b) Phase diagram (dependence of Y' on Y) is ellipsoidal.

of $\overline{n_e}$ (buildup of negative space charge). The approximate relation between $\overline{n_e}$ and ω_{pe} for $\overline{n_e}/n_o<6$ is logarithmic. Thus, for $\overline{n_e}/n_o\sim 6$, there is only a 15% increase in ω_{pe} [Fig. 1(a)]. The nonlinear relation between ω_{pe} and $\overline{n_e}$ reaches a saturation for $\overline{n_e}/n_o\geqslant 6$, since ω_{pe} has an upper limit of $\sqrt{2}\omega_{po}$.[12] The analytic formula (14) reproduces the numerically derived curve with an accuracy of 0.5% for the range $1\leqslant\overline{n_e}/n_o<6$. These results within the cold plasma fluid approximation may serve as a guide for further analysis of nonlinear cylindrical oscillations within the kinetic approach.

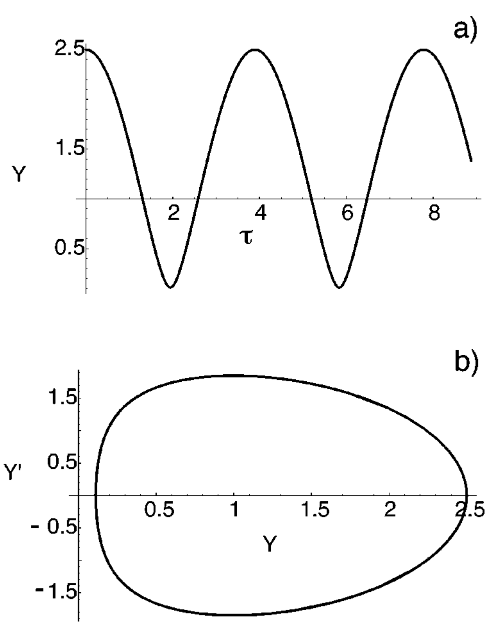

FIG. 3. (a) Numerical solution for $a=1.5$ is not sinusoidal. (b) Phase diagram shows substantial deviation from ellipse due to increased level of nonlinearity.

APPENDIX: WEAKLY NONLINEAR CASE

According to Ref. 14, the shift in frequency ω for the weakly nonlinear case $a\ll 1$ can be calculated analytically by expanding the force

$$F(Y)=-Y+Y^{-1}\approx F(Y_{eq})+F'(Y_{eq})\Delta Y + \frac{F''(Y_{eq})}{2}\Delta Y^2+\frac{F'''(Y_{eq})}{6}\Delta Y^3 \tag{A1}$$

in the basic Eq. (5) to the third order of amplitude ΔY defined as the deviation from the equilibrium point $Y_{eq}=1$:

$$Y=1+\Delta Y. \tag{A2}$$

The resulting nonlinear oscillator equation in the canonical form is

$$\Delta Y''+\omega_o^2\Delta Y=-\alpha\Delta Y^2-\beta\Delta Y^3, \tag{A3}$$

where

$$\omega_o^2=2, \tag{A4}$$

$$\alpha=-1, \tag{A5}$$

and

$$\beta=1. \tag{A6}$$

According to Ref. 15

$$Y=Y_o+Y^{(1)}+Y^{(2)}+\cdots, \tag{A7}$$

$$\omega=\omega_o+\omega^{(1)}+\omega^{(2)}+\cdots, \tag{A8}$$

where for our case $Y_o=1$, $\omega_o=\sqrt{2}$, $\omega^{(1)}=0$, $Y^{(1)}=a\cos\omega t$ and

$$\omega^{(2)}=\left(\frac{3\beta}{8\omega_o}-\frac{5\alpha^2}{12\omega_o^2}\right)a, \tag{A9}$$

$$Y^{(2)}=\frac{\alpha a^2}{2\omega_o^2}+\frac{\alpha a^2}{6\omega_o^2}\cos 2\omega t. \tag{A10}$$

Substituting Eqs. (A4)–(A6) into Eqs. (A9) and (A10), we find

$$\omega^{(2)}=\frac{\omega_o}{12}a^2, \tag{A11}$$

$$Y^{(2)}=\frac{a^2}{4}-\frac{a^2}{12}\cos 2\omega t. \tag{A12}$$

Therefore Eqs. (A7) and (A8) give

$$Y=1+\frac{a^2}{4}+a\cos\omega t-\frac{a^2}{12}\cos 2\omega t, \tag{A13}$$

$$\omega=\sqrt{2}\left(1+\frac{a^2}{12}\right). \tag{A14}$$

Equation (A14) coincides with Eq. (16) noting that $\omega_o=\sqrt{2}$ corresponds to the normalized Langmuir frequency

ω_{po}. Equation (A13) gives the explicit relation between amplitudes of the fundamental (a) and the second harmonic ($a^2/12$) for weakly nonlinear oscillations in addition to the shift of Y from unity.

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